



Supereulerianity of k -edge-connected graphs with a restriction on small bonds

Zhaohong Niu^a, Liming Xiong^{a,b,*}

^a Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China

^b Department of Mathematics, Jiangxi Normal University, Nanchang, 330022, PR China

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ABSTRACT

Let $k \geq 1$, $l > 0$, $m \geq 0$ be integers, and let $C_k(l, m)$ denote the graph family such that a graph G of order n is in $C_k(l, m)$ if and only if G is k -edge-connected such that for every bond $S \subset E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(n - m)/l$. In this paper, we show that if $G \in C_3(10, m)$ with $n > 11m$, then either G is supereulerian or it is contractible to the Petersen graph. A graph is s -supereulerian if it has a spanning even subgraph with at most s components. We also prove the following: if $G \in C_3(l, m)$ with $n > (l + 1)m$ and $l \geq 10$, then G is $\lceil (l - 4)/2 \rceil$ -supereulerian; if $G \in C_2(l, 0)$ with $6 \leq l \leq 10$, then G is $(l - 4)$ -supereulerian; if $G \in C_2(l, m)$ with $n > (l + 1)m$ and $l \geq 4$, then G is $(l - 3)$ -supereulerian.

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1. Introduction

We use [1] for terminology and notation not defined here and consider only finite simple graphs, unless otherwise stated.

A *bond* of G is a minimal nonempty edge cut. A graph is called *trivial* if it has only one vertex; *nontrivial* otherwise. Let $O(G)$ denote the set of all odd degree vertices of G . A graph G with $O(G) = \emptyset$ is an *even graph*, and a connected even graph is an *eulerian graph*. A graph is *supereulerian* if it has a spanning eulerian subgraph. We use \mathcal{SL} to denote the families of supereulerian graphs. Regard K_1 as supereulerian. A graph G is called *s-supereulerian* if G has a spanning even subgraph with at most s components. Jaeger [9] showed the following theorem:

Theorem 1 (Jaeger, [9]). *Every 4-edge-connected graph is supereulerian.*

Let $k \geq 1$, $l > 0$, $m \geq 0$ be integers; let $C_k(l, m)$ denote the graph family such that a graph G of order n is in $C_k(l, m)$ if and only if G is k -edge-connected such that for every bond $S \subset E(G)$ with $|S| \leq 3$, each component of $G - S$ has order at least $(n - m)/l$.

By the definition of $C_k(l, m)$, we have the following proposition immediately.

Proposition 2. *Let $k, k_1, k_2 \geq 1$, $l, l_1, l_2 > 0$ and $m, m_1, m_2 \geq 0$ be integers. Then:*

- (a) *If $k_1 < k_2$, then $C_{k_1}(l, m) \supset C_{k_2}(l, m)$.*
- (b) *If $l_1 < l_2$, then $C_k(l_1, m) \subset C_k(l_2, m)$.*
- (c) *If $m_1 < m_2$, then $C_k(l, m_1) \subset C_k(l, m_2)$.*

* Corresponding author at: Department of Mathematics, Beijing Institute of Technology, Beijing, PR China.
E-mail address: lmxiong@bit.edu.cn (L. Xiong).

Note that every 4-edge-connected graph has no bond $S \subset E(G)$ with $|S| \leq 3$. If $k \geq 4$, $C_k(l, m)$ consists of all k -edge-connected graphs. Hence supereulerianity in that case has already been proven by [Theorem 1](#). So there remain the cases when $k \leq 3$. In this paper, we consider supereulerianity of the graphs in $C_3(l, m)$ and $C_2(l, m)$.

For $k = 3$, we will prove in [Theorem 11](#) that if $G \in C_3(10, m)$ with $n > 11m$, then either G is supereulerian or it is contractible to the Petersen graph. We also extend this result to s -supereulerian graphs.

For $k = 2$, Catlin and Li, Broersma and Xiong, Li, Lai and Zhan have proved some results concerning when a graph in a certain family $C_2(l, m)$ is supereulerian; see [\[7,2,10\]](#), respectively. More recently, Li et al. [\[11\]](#) showed that if $G \in C_2(6, m)$ is a graph with $n = |V(G)| > 7m$, then G is supereulerian if and only if G cannot be contracted to some well classified special graphs (we do not describe them in detail here since they play no role in the rest of the paper). In this paper, we extend these results to graphs in $C_2(l, m)$ for large l , and show that if $G \in C_2(l, 0)$ with $6 \leq l \leq 10$, then G is $(l - 4)$ -supereulerian; if $G \in C_2(l, m)$ with $n = |V(G)| > (l + 1)m$ and $l \geq 4$, then G is $(l - 3)$ -supereulerian.

In [Section 2](#), we shall present Catlin's reduction method and some previous results. In [Section 3](#), we prove an auxiliary result, which will be used in the proof of the main results. The main results on graphs in $C_3(l, m)$ and $C_2(l, m)$ will be proved in [Sections 4](#) and [5](#), respectively. [Section 6](#) will be devoted to some remarks and open problems relating to the bounds of the main results.

2. Preliminaries

For a graph G with a connected subgraph H , the contraction G/H is the multigraph obtained from G by replacing H by a new vertex v_H , such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H . v_H is called the image of H .

A graph H is called collapsible if for every even set $X \subseteq V(H)$, there is a spanning connected subgraph H_X of H such that $O(H_X) = X$.

In [\[3\]](#), Catlin showed that any graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs H_1, H_2, \dots, H_c such that $\bigcup_{i=1}^c V(H_i) = V(G)$. The reduction of G , denoted by G' , is the graph obtained from G by contracting each H_i ($1 \leq i \leq c$) to a single vertex. A graph G is reduced if $G = G'$ (see [\[3\]](#)). For surveys of work on supereulerian graphs, see [\[4,8\]](#).

In [\[3\]](#), Catlin proved the following theorem:

Theorem 3 (Catlin, [\[3\]](#)). Let G be a connected graph and G' be the reduction of G . Then:

- (a) $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$;
- (b) for any subgraph H of G' , either $H \in \{K_1, K_2\}$ or $|E(H)| \leq 2|V(H)| - 4$;
- (c) G' is simple and has no K_3 .

For a graph G , define $F(G)$ to be the minimum number of edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 4 (Catlin, Han and Lai, [\[5\]](#)). If G is reduced with $|V(G)| \geq 3$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

Theorem 5 (Catlin, Han and Lai, [\[5\]](#)). Let G be a 2-edge-connected reduced graph. If $F(G) \leq 2$, then $G \in \mathcal{SL}$ or $G \cong K_{2,t}$, where t is odd.

Let $D_i(G) = \{v \in V(G) \mid d(v) = i\}$ and $d_i(G) = |D_i(G)|$.

Theorem 6 (Catlin, [\[3\]](#)). If G is a nontrivial 2-edge-connected reduced graph, then $d_2(G) + d_3(G) \geq 4$. If $d_2(G) + d_3(G) = 4$, then G is eulerian and G has four vertices of degree 2.

Throughout the whole paper, we let P denote the Petersen graph, which is the smallest 3-edge-connected nonsupereulerian reduced graph. For 3-edge-connected graphs, Catlin and Lai [\[6\]](#) proved the following two results:

Theorem 7 (Catlin and Lai, [\[6\]](#)). Let G be a 3-edge-connected graph and G' be the reduction of G . If G has at most ten edge cuts of size 3, then either $G \in \mathcal{SL}$ or $G' = P$.

Theorem 8 (Catlin and Lai, [\[6\]](#)). Let G be a 3-edge-connected reduced graph with $F(G) = 3$, then either $G \in \mathcal{SL}$ or G has no 2-edge-connected subgraph H with $F(H) = 2$.

3. Two auxiliary results

Lemma 9. Let $G \in C_2(l, m)$ be a graph with $n = |V(G)| > (l + 1)m$. Then either $G' = K_1$ or $d_2(G') + d_3(G') \leq l$, where G' is the reduction of G .

Proof. Suppose $G' \neq K_1$ and $d_2(G') + d_3(G') \geq l + 1$. Since $G' \neq K_1$, we can assume that G' is 2-edge-connected and nontrivial.

Let $c = d_2(G') + d_3(G')$, and H_1, H_2, \dots, H_c denote the subgraphs of G whose contraction images in G' are the vertices of degree at most 3 in G' . Since $G \in C_2(l, m)$, for each i with $1 \leq i \leq c$, $|V(H_i)| \geq (n - m)/l$. It follows since $c \geq l + 1$ that

$$n = |V(G)| \geq \sum_{i=1}^c |V(H_i)| \geq \frac{c(n - m)}{l} \geq \frac{(l + 1)(n - m)}{l},$$

and so $ln \geq (l + 1)n - (l + 1)m$, contrary to the assumption that $n > (l + 1)m$. This completes the proof. \square

Lemma 10. Let G be a 2-edge-connected reduced graph and $d_i = |d_i(G)|$. Then

$$2F(G) + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3.$$

Proof. Since G is a 2-edge-connected reduced graph, by Theorem 3(c), we have $|V(G)| \geq 4$. Hence by Theorem 4,

$$2F(G) = 4|V(G)| - 2|E(G)| - 4.$$

Since G is 2-edge-connected, $2|E(G)| = \sum_{j \geq 2} jd_j$ and $|V(G)| = \sum_{j \geq 2} d_j$, and so

$$2F(G) + 4 = \sum_{j \geq 2} 4d_j - \sum_{j \geq 2} jd_j = \sum_{j \geq 2} (4 - j)d_j = 2d_2 + d_3 + \sum_{j \geq 5} (4 - j)d_j.$$

Hence we have $2F(G) + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3$. \square

4. 3-edge-connected graphs

In this section, we consider the graphs in $C_3(l, m)$.

Theorem 11. Let $m \geq 0$ be an integer and $G \in C_3(10, m)$ be a graph with $n = |V(G)| > 11m$. Then either $G \in \mathcal{SL}$ or the reduction of G is the Petersen graph.

Proof. Let G' be the reduction of G . By Theorem 3(a), it suffices to show $G' \in \mathcal{SL}$ or $G' = P$. Since $K_1 \in \mathcal{SL}$, if $G' = K_1$, then we are done, so we may assume that G' is 3-edge-connected and nontrivial. Let $d_i = |D_i(G')|$, and we have

$$d_1 = d_2 = 0, \quad d_i \geq 0, \quad \text{for } i \geq 3. \quad (1)$$

We shall assume that

$$G' \notin \mathcal{SL} \quad \text{and} \quad G' \neq P \quad (2)$$

to find a contradiction.

If $F(G') \leq 2$, by Theorem 5 and by (2), $G' \cong K_{2,t}$, with t being odd, which will yield vertices of degree 2 and contrary to (1). By Theorem 6, if $d_2 + d_3 = 4$, then $G' \in \mathcal{SL}$, contrary to (2). By Proposition 2(a), $C_3(10, m) \subset C_2(10, m)$. So by Lemma 9 and since G' is nontrivial, $d_2 + d_3 \leq 10$. Therefore, we only consider the case when $F(G') \geq 3$ and $5 \leq d_3 \leq 10$ (by $5 \leq d_2 + d_3 \leq 10$ and (1)).

From Lemma 10 and since $d_2 = 0$ (by (1)), $d_3 \leq 10$, and $F(G') \geq 3$, we have

$$10 + \sum_{j \geq 5} (j - 4)d_j \leq 2F(G) + 4 + \sum_{j \geq 5} (j - 4)d_j = d_3 \leq 10.$$

So all terms here are equal, implying that $d_3 = 10$, $d_i = 0$ for $i \geq 5$, and $F(G') = 3$.

Since $d_3 = 10$, G' has at least ten edge cuts of size 3. If G' has exactly ten edge cuts of size 3, by Theorem 7, either $G' \in \mathcal{SL}$ or $G' = P$, contrary to (2). So we may assume there exists at least one edge cut S' of size 3 such that the two components of $G' - S'$, denoted by H_1 and H_2 , are nontrivial.

First we prove the following two claims.

Claim 1. H_i is 2-edge-connected, for $i = 1, 2$.

Proof of Claim 1. Suppose at least one of $\{H_1, H_2\}$, say H_1 , has a cut edge e , and let W_1, W_2 be the two components of $H_1 - e$. Let V' denote the set of vertices of $V(H_1)$ incident with at least one edge of S' , and $m_j = |V(W_j) \cap V'|$, for $j = 1, 2$. Then $(V(W_1) \cap V') \cup (V(W_2) \cap V') = V'$, and hence $m_1 + m_2 = |V'|$. Since $|S'| = 3$ and H_1, H_2 are the two components of $G' - S'$, we have $|V'| \leq 3$, and then $m_1 + m_2 \leq 3$. Without loss of generality, we can assume that $m_1 \leq m_2$; then $m_1 \leq 1$.

If $m_1 = 0$, i.e., three edges of S' are all incident with vertices in W_2 , then e is a cut edge of G' , contrary to the fact that G' is 3-edge-connected.

If $m_1 = 1$, i.e., exactly one edge e' (say) of S' is incident with a vertex of W_1 , then $\{e, e'\}$ is an edge cut (of size 2) of G' , contrary to the fact that G' is 3-edge-connected.

Hence H_1 has no cut edge, i.e., H_1 is 2-edge-connected. So Claim 1 holds. \square

Let $d_3^*(H_i) = |D_3(G') \cap V(H_i)|$, and $d_4^*(H_i) = |D_4(G') \cap V(H_i)|$, for $i = 1, 2$. Then $|V(H_i)| = d_3^*(H_i) + d_4^*(H_i)$, and $d_3^*(H_1) + d_3^*(H_2) = d_3 = 10$.

Claim 2. $d_3^*(H_i) = 5$, for $i = 1, 2$.

Proof of Claim 2. Since G' is reduced, by Claim 1 and by Theorem 3(b), $|E(H_i)| \leq 2|V(H_i)| - 4$. Note that

$$2|E(H_i)| = 3d_3^*(H_i) + 4d_4^*(H_i) - 3, \quad \text{and} \quad |V(H_i)| = d_3^*(H_i) + d_4^*(H_i), \quad (3)$$

by (3),

$$3d_3^*(H_i) + 4d_4^*(H_i) - 3 \leq 4(d_3^*(H_i) + d_4^*(H_i)) - 8,$$

and hence $d_3^*(H_i) \geq 5$. By $d_3^*(H_1) + d_3^*(H_2) = 10$, we have $d_3^*(H_i) = 5$ for $i = 1, 2$; Claim 2 holds. \square

For $i = 1, 2$, since H_i is reduced, and $|V(H_i)| \geq d_3^*(H_i) = 5 > 3$, by Theorem 4, $F(H_i) = 2|V(H_i)| - |E(H_i)| - 2$.

By (3) and Claim 2, we have

$$\begin{aligned} 2F(H_i) &= 4|V(H_i)| - 2|E(H_i)| - 4 \\ &= 4 \times (d_3^*(H_i) + d_4^*(H_i)) - (3d_3^*(H_i) + 4d_4^*(H_i) - 3) - 4 \\ &= d_3^*(H_i) - 1 \\ &= 4 \end{aligned}$$

and then $F(H_i) = 2$, for $i = 1, 2$, which contradicts Theorem 8 since $G' \notin \mathcal{SL}$ and H_i is 2-edge-connected (by Claim 1).

This completes the proof of Theorem 11. \square

In [12], Niu et al. extend Theorem 3 and another theorem of Catlin [3] to the following results:

Theorem 12 (Niu, Lai and Xiong, [12]). Let G be a connected graph and G' be the reduction of G . Then G is s -supereulerian if and only if G' is s -supereulerian.

Theorem 13 (Niu, Lai and Xiong, [12]). Let G be a 2-edge-connected graph. If $F(G) \leq s$, then G is s -supereulerian.

For $l \geq 10$, we prove the following result, which is slightly weaker than Theorem 11 when $l = 10$.

Theorem 14. Let $l \geq 10$, $m \geq 0$ be two integers and $G \in C_3(l, m)$ be a graph with $n = |V(G)| > (l + 1)m$; then G is $\lceil \frac{l-4}{2} \rceil$ -supereulerian.

Proof. Let G' be the reduction of G and $d_i = |D_i(G')|$. By Theorem 12, it suffices to show G' is $\lceil (l - 4)/2 \rceil$ -supereulerian. Arguing like in the proof of Theorem 11, it suffices to prove the case when G' is 3-edge-connected and nontrivial and $5 \leq d_3 \leq l$.

Since G' is 3-edge-connected, if $F(G') \leq \lceil (l - 4)/2 \rceil$, by Theorem 13, G' is $\lceil (l - 4)/2 \rceil$ -supereulerian; Theorem 14 holds. So we can assume $F(G') > \lceil (l - 4)/2 \rceil$. From Lemma 10 and since $d_2 = 0$, $d_3 \leq l$, we have

$$l \leq 2 \left\lceil \frac{l-4}{2} \right\rceil + 4 < 2F(G') + 4 + \sum_{j \geq 5} (j-4)d_j = d_3 \leq l,$$

a contradiction.

This completes the proof of Theorem 14. \square

5. 2-edge-connected graphs

In this section, we consider the graphs in $C_2(l, 0)$ ($6 \leq l \leq 10$) and $C_2(l, m)$ ($l \geq 7$).

Theorem 15. Let $6 \leq l \leq 10$ be an integer and $G \in C_2(l, 0)$ be a graph of order n ; then G is $(l - 4)$ -supereulerian.

The following theorem will be used in the proof of Theorem 15.

Theorem 16 (Niu and Xiong, [13]). Let G be a 2-edge-connected reduced graph of order n and $s \in \{1, 2, 3\}$ such that $n \leq 3s + 1$; then G is s -supereulerian.

Now we prove Theorem 15.

Proof of Theorem 15. Let G' be the reduction of G ; by Theorem 12, it suffices to show G' is $(l - 4)$ -supereulerian. Since $K_1 \in \mathcal{SL}$, if $G' = K_1$, then we are done, so we may assume that G' is 2-edge-connected and nontrivial. Let $d_i = |D_i(G')|$.

By Theorem 6, if $d_2 + d_3 = 4$, then $G' \in \mathcal{SL}$; by Lemma 9, $d_2 + d_3 \leq l$. Therefore, we only consider the case when $5 \leq d_2 + d_3 \leq l$. We shall assume that

$$G' \text{ is not } (l - 4)\text{-supereulerian}, \quad (4)$$

to find a contradiction.

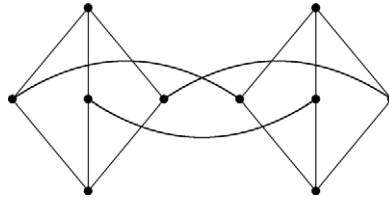


Fig. 1. G_1 .

Case 1. $5 \leq d_2 + d_3 \leq l - 1$.

If $F(G') \leq l - 4$, by Theorem 13, G' is $(l - 4)$ -supereulerian, contrary to (4). So we may assume that $F(G') \geq l - 3$. From Lemma 10 and since $d_2 + d_3 \leq l - 1$, we have

$$2(l - 1) + \sum_{j \geq 5} (j - 4)d_j \leq 2F(G') + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3 \leq d_2 + (l - 1) \leq 2(l - 1).$$

So all terms here are equal, implying that $d_2 = l - 1$, $d_3 = 0$, and $d_j = 0$ ($j \geq 5$). Thus G' is eulerian, and so $G' \in \mathcal{EL}$, contrary to (4).

Case 2. $d_2 + d_3 = l$.

Let H_1, H_2, \dots, H_l denote the subgraphs of G whose contraction images in G' are the vertices of degree at most 3 in G' . Since $G \in C_2(l, 0)$, for each i with $1 \leq i \leq l$, $|V(H_i)| \geq n/l$. It follows that

$$n = |V(G)| \geq \sum_{i=1}^l |V(H_i)| \geq \frac{ln}{l} = n,$$

and hence $|V(G')| = l$. If $l = 6, 7$, by Theorem 16, G' is 2-supereulerian, and hence G' is $(l - 4)$ -supereulerian, contrary to (4); if $8 \leq l \leq 10$, by Theorem 16, G' is 3-supereulerian, and hence G' is $(l - 4)$ -supereulerian, contrary to (4).

This completes the proof of Theorem 15. \square

If $G \in C_2(l, m)$, we can get the following result:

Theorem 17. Let $l \geq 7$, $m \geq 0$ be two integers and $G \in C_2(l, m)$ be a graph with $n = |V(G)| > (l + 1)m$; then G is $(l - 3)$ -supereulerian.

Proof. Let G' be the reduction of G ; by Theorem 12, it suffices to show G' is $(l - 3)$ -supereulerian. Since $K_1 \in \mathcal{EL}$, if $G' = K_1$, then we are done, so we may assume that G' is 2-edge-connected and nontrivial. Let $d_i = |D_i(G')|$ and $c = d_2 + d_3$.

If $c = 4$, then $G' \in \mathcal{EL}$ by Theorem 6. Hence by Lemma 9, it remains to consider the case when $5 \leq c \leq l$. We shall assume that

$$G' \text{ is not } (l - 3)\text{-supereulerian} \tag{5}$$

to find a contradiction.

If $F(G') \leq c - 3$ ($\leq l - 3$), by Theorem 13, G' is $(l - 3)$ -supereulerian, contrary to (5). So we may assume that $F(G') \geq c - 2$. Then by Lemma 10 and since $d_2 + d_3 = c \leq l$, we have

$$2c + \sum_{j \geq 5} (j - 4)d_j \leq 2F(G') + 4 + \sum_{j \geq 5} (j - 4)d_j = 2d_2 + d_3 \leq c + d_2 \leq 2c.$$

So all terms here are equal, implying that $d_2 = c$, $d_3 = 0$ and $d_j = 0$ ($j \geq 5$). Thus G' is eulerian, and so $G' \in \mathcal{EL}$, contrary to (5).

This completes the proof of Theorem 17. \square

6. Remarks and open problems

First we show that there exists a supereulerian graph which satisfies the hypotheses of Theorem 11 but doesn't satisfy the hypotheses of Theorem 7: in graph G_1 , depicted in Fig. 1, each solid vertex represents a complete graph of order s , where $s \geq 5$ is an integer; then $G_1 \in C_3(10, 0)$. Since the reduction of G_1 is not the Petersen graph, by Theorem 11, G is supereulerian, but this cannot be deduced by Theorem 7 since G_1 has eleven edge cuts of size 3.

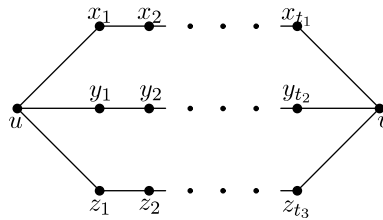
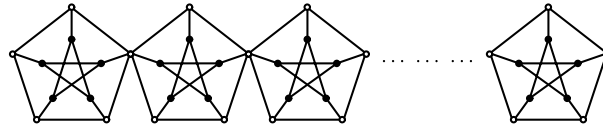
Theorems 1, 11, 14 and 17 can be summarized to the following problem:

For integers $k \geq 1$, $l > 0$ and $m \geq 0$, determine the minimum integer $f_k(l, m)$ for which the statement

Let $G \in C_k(l, m)$ be a graph with $n = |V(G)| > (l + 1)m$; then G is $f_k(l, m)$ -supereulerian.

is valid.

For $k \geq 4$, Theorem 1 shows that $f_k(l, 0) = 1$.

Fig. 2. $K_{2,3}(t_1, t_2, t_3)$.Fig. 3. $G(t)$.

For $k = 3$, Theorem 11 shows that $f_3(10, m) = 2$ (note that the Petersen graph is 2-supereulerian) and $f_3(l, m) = 1$ for $l \leq 9$. Theorem 14 shows that if $l \geq 10$, then $f_3(l, m) \leq \lceil (l-4)/2 \rceil$ (this inequality also holds for $5 \leq l \leq 9$ since $f_3(l, m) = 1$ for $l \leq 9$).

For $k = 2$, Theorem 17 shows that if $l \geq 7$, then $f_2(l, m) \leq l - 3$. If $4 \leq l \leq 6$, the corresponding results in [7,2,10,11] imply that $f_2(l, m) \leq l - 3$ also holds. So we have $f_2(l, m) \leq l - 3$ for $l \geq 4$.

In the following, we construct some special graphs implying that $f_2(l, m) \geq \lfloor (l+1)/3 \rfloor$ and $f_3(l, m) \geq \lfloor (l+5)/8 \rfloor$.

Let $0 < t_1 \leq t_2 \leq t_3$ be integers, u, v the vertices of $K_{2,3}$ with degree 3, and $K_{2,3}(t_1, t_2, t_3)$ a graph obtained from $K_{2,3}$ by replacing each $u - v$ path by a path of length $t_i + 1$ ($i = 1, 2, 3$), depicted in Fig. 2. Obviously, $K_{2,3}(1, 1, 1) \cong K_{2,3}$, and

$K_{2,3}(t_1, t_2, t_3)$ is $(t_1 + 1)$ -supereulerian, but not t_1 -supereulerian. (6)

For a positive integer s , the s -enlarging of a vertex v of G is the graph $G_s(v)$ obtained from G by replacing v by a complete graph of order s . Note that $G_s(v)' = G'$ and by Theorem 12, when we consider the supereulerianity of $G_s(v)$, it does not matter which vertices of the complete graph of order s are incident with the edges from the original graph G .

Let $t > 0, s \geq 5, m \geq 0$ be integers; obtain the graph G_2 from $K_{2,3}(t, t, t)$ by s -enlarging every vertex of $K_{2,3}(t, t, t)$. Then $G_2' = K_{2,3}(t, t, t)$, $n(G_2) = (3t + 2) \cdot s$, and $d_2(G_2') + d_3(G_2') = 3t + 2$. Since $s \geq 5$ and by Proposition 2(c), $G_2 \in C_2(3t + 2, 0) \subseteq C_2(3t + 2, m)$. By $G_2' = K_{2,3}(t, t, t)$, (6) and Theorem 12, G_2 is $(t + 1)$ -supereulerian, but not t -supereulerian. Since the condition $|V(G_2)| \geq ((3t + 2) + 1) \cdot m$ always holds for given t and m by taking s , and hence n , to be sufficiently large, say $s \geq (3t + 3) \cdot m / (3t + 2)$, and by the definition of $f_k(l, m)$, we have $f_2(3t + 2, m) \geq t + 1$. Hence $f_2(l, m) \geq (l + 1)/3$ for $l = 3t + 2$.

By Proposition 2(c) and (b), $G_2 \in C_2(3t + 2, m) \subseteq C_2(3t + 3, m) \subseteq C_2(3t + 4, m)$, we can take l to be $3t + 3$ or $3t + 4$ as well as $3t + 2$ and have a graph $G_2 \in C_2(l, m)$ that is $(t + 1)$ -supereulerian, but not t -supereulerian. Then by the definition of $f_k(l, m)$, we have $f_2(3t + i, m) \geq t + 1$ for $i = 2, 3, 4$ and $t > 0$. Hence $f_2(l, m) \geq \lfloor (l + 1)/3 \rfloor$.

Let t be a positive integer, and let $G(t)$ be the graph obtained from t Petersen graphs, depicted in Fig. 3. Then $G(t)$ has a spanning even subgraph with $t + 1$ components: the subgraph induced by the solid vertices is an even graph with t components and the subgraph induced by the hollow vertices is an eulerian subgraph. By the induction method on t , one can prove that any spanning even subgraph of $G(t)$ has at least $t + 1$ components. So

$G(t)$ is $(t + 1)$ -supereulerian, but not t -supereulerian. (7)

Let $s \geq 5, m \geq 0$ be integers; obtain the graph $G_3(t)$ from $G(t)$ by s -enlarging every vertex of degree 3 in $G(t)$. Then $G_3(t)' = G(t)$, $n(G_3(t)) = (8t + 2) \cdot s + (t - 1)$, and $d_3(G_3(t)') = 8t + 2$. Since $s \geq 5$ and by Proposition 2(c), $G_3(t) \in C_3(8t + 2, t - 1) \subseteq C_3(8t + 2, m)$ when $m \geq t - 1$. By $G_3(t)' = G(t)$, (7) and Theorem 12, $G_3(t)$ is $(t + 1)$ -supereulerian, but not t -supereulerian. Since the condition $|V(G_3(t))| \geq ((8t + 2) + 1) \cdot m$ always holds for given t and m by taking s , and hence n , to be sufficiently large, say $s \geq ((8t + 3) \cdot m - (t - 1)) / (8t + 2)$, and by the definition of $f_k(l, m)$, we have $f_3(8t + 2, m) \geq t + 1$ for $m \geq t - 1$. Hence $f_3(l, m) \geq (l + 6)/8$ for $l = 8t + 2$ and $m \geq (l - 10)/8$.

By Proposition 2(c) and (b), $G_3(t) \in C_3(8t + 2, m) \subseteq C_3(8t + i, m)$ when $m \geq t - 1$ and $3 \leq i \leq 9$, we can take l to be $8t + i$ as well as $8t + 2$ and have a graph $G_3(t) \in C_3(l, m)$ that is $(t + 1)$ -supereulerian, but not t -supereulerian. Then by the definition of $f_k(l, m)$, we have $f_3(8t + i, m) \geq t + 1$ for $2 \leq i \leq 9, t > 0$ and $m \geq t - 1$. Hence $f_3(l, m) \geq \lfloor (l + 6)/8 \rfloor$ for $m \geq \lfloor (l - 10)/8 \rfloor$.

To get a lower bound on $f_3(l, m)$ that is valid for all m , note that if $s \geq t - 1$, we will have $(8t + 3)s \geq (8t + 2)s + (t - 1) = n$, and so $s \geq n / (8t + 3)$ and $G_3(t) \in C_3(8t + 3, 0) \subseteq C_3(8t + 3, m)$ for all m by Proposition 2(c). From this and arguing like in the above paragraph, we get that $f_3(l, m) \geq \lfloor (l + 5)/8 \rfloor$.

Therefore, we know that $\lfloor (l + 1)/3 \rfloor \leq f_2(l, m) \leq l - 3$ for $l \geq 4$, and $\lfloor (l + 6)/8 \rfloor \leq f_3(l, m) \leq \lceil (l - 4)/2 \rceil$ for $l \geq 5$ and $m \geq \lfloor (l - 10)/8 \rfloor$, and $\lfloor (l + 5)/8 \rfloor \leq f_3(l, m) \leq \lceil (l - 4)/2 \rceil$ for $l \geq 5$ and all m , but the exact values of $f_2(l, m)$ and $f_3(l, m)$ remain to be determined.

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